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A note on interior vs. boundary-layer damping of surface waves in a circular cylinder

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Martel *et al.* (1998) have shown that interior damping may be comparable with boundary-layer damping for surface waves in small cylinders and that its incorporation yields predictions in agreement with the experimental results of Henderson & Miles (1994) for non-axisymmetric waves on a clean surface with a fixed contact line. In the present note, Henderson & Miles's boundary-layer calculation is supplemented by a calculation of interior damping based on Lamb's dissipation integral for an irrotational flow. The analysis, which omits second-order boundary-layer effects, is simpler than that of Martel *et al.* (which includes these effects and is based on an expansion in an inverse Reynolds number), but yields results of comparable accuracy within the parametric domain of the experiments. The corresponding calculations for a fully contaminated (inextensible) surface reduce the discrepancy between calculation and experiment but, in contrast to the results for a clean surface, leave a significant residual discrepancy. An unexplained discrepancy also remains for axisymmetric waves on either a clean or a contaminated surface.

Case & Parkinson (1957) and Martel, Nicolás & Vega (1998) have remarked that, although the contributions of boundary-layer and interior damping of surface waves in a cylinder of lateral scale a are proportional to l_v/a and $(l_v/a)^2$, respectively, where

$$l_v \equiv (2v/\omega)^{1/2} \ll a \tag{1}$$

is the viscous length (v = kinematic viscosity, ω = angular frequency), these contributions may be of comparable magnitude. In particular, Martel *et al.* have shown that the incorporation of interior damping resolves the discrepancy between Henderson & Miles's (1994, hereinafter referred to as HM) boundary-layer approximation to, and measurements of, the damping of non-axisymmetric waves (but a significant discrepancy remains for the dominant axisymmetric mode) in a circular cylinder with a fixed contact line. Martel *et al.* develop the solution of the boundary-value problem in powers of a parameter $C^{1/2}$, which is proportional to l_v/a. Their solution comprises the $O(1)$ inviscid solution, the $O(C^{1/2})$ boundary-layer contribution, and $O(C)$ contributions to the total damping from both interior dissipation and the boundary layers.

A simpler, although somewhat less accurate (see below), procedure for the calculation of the damping for $l_v \ll a$ is to use the inviscid solution to evaluate the dominant (for $l_v/a \downarrow 0$) terms in the boundary-layer and interior-dissipation integrals. The resulting omission of the $O(C)$ boundary-layer term would be inconsistent with the retention of the $O(C)$ interior damping if these two terms had equal standing in an asymptotic expansion, but, in our view, they may be regarded as physically independent at the present level of approximation; indeed, it appears to us that a major contribution of Martel *et al.*'s paper is to establish that the $O(C)$ interior damping may be important even though the $O(C)$ boundary-layer damping is negligible.

Against this background, we have supplemented our boundary-layer-dissipation-rate integral (HM, § 3) for a circular cylinder of radius a and depth d with Lamb's (1932, § 329(13)) integral

$$D = -\mu \iint \partial_n (\nabla\phi)^2 \mathrm{d}S \tag{2}$$

for the dissipation rate for an irrotational fluid motion in a fluid of boundary S, which, in general, comprises both rigid and free-surface parts ($\mu \equiv \rho\nu$ is the viscosity, ϕ is the velocity potential, and n is the inwardly directed normal to S). The integrand in (2) vanishes on the bottom ($z = -d$, where $\partial\phi/\partial n = \partial\phi/\partial z = 0$), which therefore makes no contribution to D. The lateral boundary ($r = a$, where $\partial\phi/\partial n = -\partial\phi/\partial r = 0$) contributes†

$$\frac{D_c}{\mu} = \iint \partial_r \left(\frac{1}{r^2}\phi_\theta^2\right) r\mathrm{d}r\mathrm{d}\theta = -\frac{2}{a^2}\iint (\phi_\theta^2)_{r=a}\mathrm{d}\theta\,\mathrm{d}z\ . \tag{3}$$

The free surface ($z \simeq 0$, where $\partial\phi/\partial n = -\partial\phi/\partial z$) contributes

$$\frac{D_S}{\mu} = \iint \partial_z \left(\phi_r^2 + \frac{1}{r^2}\phi_\theta^2 + \phi_z^2\right) r\mathrm{d}r\mathrm{d}\theta \tag{4a}$$

$$= 4\iint (\phi_z\phi_{zz})_{z=0}\, r\mathrm{d}r\mathrm{d}\theta, \tag{4b}$$

where (4 *b*) follows from (4 *a*) through Laplace's equation for ϕ, Green's theorem, and the boundary condition $\partial\phi/\partial r = 0$ at the lateral boundary.

We pose the velocity potential for free oscillations in the form (HM, (2.4))

$$\phi = \phi_n(t)R_n(r)\cos s\theta \frac{\cosh k_n(z+d)}{\cosh k_n d}, \tag{5}$$

where

$$R_n(r) = \frac{J_s(k_n r)}{J_s(k_n a)}, \quad J_s'(\kappa_n) = 0, \quad \kappa_n \equiv k_n a, \tag{6a,b}$$

s is the azimuthal wavenumber, J_s is a Bessel function, and, here and subsequently, repeated indices are summed over the complete, orthogonal set $\{R_n, k_n\}$ except where the index occurs once but is not repeated on one side of an equation. Note that R_n and k_n depend on s and that $\{1, 0\}$ is a non-trivial member of $\{R_n, k_n\}$ if and only if

† The damping rate D_c is associated with the curvature (hence the subscript c) of the lateral boundary and would vanish for a plane wall. The present D_c and D_S are proportional to the first and second integrals on the right-hand side of Martel *et al.*'s (2.29 *a*). We had omitted D_c in an earlier reduction of (2) and are indebted to Dr Vega for pointing out our error. Case & Parkinson (1956) appear to have made a similar error in reducing their (31) to their (34). The numerical error in omitting D_c in the calculation of interior dissipation is less than $\frac{1}{2}$% for all modes considered here.

$s = 0$. Substituting (5) into (3) and (4 b), we obtain

$$\frac{D_c}{\mu} = -\frac{2\pi s^2}{a}\left(\frac{\kappa_m T_m - \kappa_n T_n}{\kappa_m^2 - \kappa_n^2}\right)\phi_m\phi_n, \quad T_n \equiv \tanh k_n d, \tag{7a,b}$$

and

$$\frac{D_S}{\mu} = \frac{2\pi}{a}(1 + \delta_{0s})\kappa_n(\kappa_n^2 - s^2)T_n\phi_n^2, \tag{8}$$

where δ_{0s} is the Kronecker delta. The corresponding inviscid approximation to the total energy (on the assumption of equal mean kinetic and potential energies for small oscillations) of the fluid motion is given by

$$\frac{E}{\rho} = \iint (\phi\phi_z)_{z=0}\, r\mathrm{d}r\mathrm{d}\theta \tag{9a}$$

$$= \tfrac{1}{2}\pi a(1 + \delta_{0s})\kappa_n^{-1}(\kappa_n^2 - s^2)T_n\phi_n^2. \tag{9b}$$

The mean values of D and E for a free oscillation of frequency ω are obtained by regarding ϕ as a complex amplitude of the carrier $\exp(i\omega t)$ and replacing ϕ_n^2 by $\frac{1}{2}|\phi_n|^2$ and $\phi_m\phi_n$ by $\frac{1}{4}(\phi_m\bar{\phi}_n + \bar{\phi}_m\phi_n)$, where $\bar{\phi}_n$ is the complex conjugate of ϕ_n. The corresponding interior-damping ratio for deep water ($T_n \simeq 1$) is

$$\delta_i = \frac{D_c + D_S}{2\omega E} = \left(\frac{l_v}{a}\right)^2\left[\frac{\kappa_n(\kappa_n^2 - s^2)|\phi_n|^2 - s^2(\kappa_m + \kappa_n)^{-1}\phi_m\bar{\phi}_n}{\kappa_l^{-1}(\kappa_l^2 - s^2)|\phi_l|^2}\right], \tag{10}$$

which must be added to the boundary-layer-damping ratio (HM, (3.10))

$$\delta_w = \frac{1}{2}\left(\frac{l_v}{a}\right)\left[\frac{(\kappa_m\kappa_n + s^2)(\kappa_m + \kappa_n)^{-1}\phi_m\bar{\phi}_n}{\kappa_l^{-1}(\kappa_\ell^2 - s^2)|\phi_\ell|^2}\right]. \tag{11}$$

The corresponding ratio of interior/wall (subscripts i/w) damping is

$$\frac{\delta_i}{\delta_w} = \frac{2l_v}{a}\left[\frac{\kappa_\ell(\kappa_\ell^2 - s^2)|\phi_\ell|^2 - s^2(\kappa_m + \kappa_n)^{-1}\phi_m\bar{\phi}_n}{(\kappa_m\kappa_n + s^2)(\kappa_m + \kappa_n)^{-1}\phi_m\bar{\phi}_n}\right] \tag{12a}$$

$$\simeq \frac{2l_v}{a}\left[\frac{\kappa_l(\kappa_l^2 - s^2)|\phi_l|^2}{(\kappa_m\kappa_n + s^2)(\kappa_m + \kappa_n)^{-1}\phi_m\bar{\phi}_n}\right], \tag{12b}$$

where (12 b) follows from (12 a) on the neglect of D_c.

Results for our clean-surface (HM, § 5.1) experiments are listed in table 1, confirming Martel *et al.*'s conclusion that the interior damping is significant for all modes considered and dominates the calculated wall damping for the (0,1) and (1,1) modes. Moreover, its inclusion essentially resolves the discrepancy between prediction and measurement except for the (0,1) mode. (We have listed the experimental/calculated damping ratio to two decimals in order to compare with Martel *et al.*, but only one decimal is justified by the experimental accuracy.) Our approximation to the damping is consistently 2–3% smaller than that of Martel *et al.*, perhaps because of our neglect of second-order boundary-layer effects, but the difference is small compared with the contribution of interior damping. The viscous-corrected natural frequencies, $f^{(3)}$, now are slightly smaller than those listed in column seven of HM, table 1, but the agreement between predicted and measured values (see below for (0,1) mode) remains within the experimental error.

We also have corrected an error in our calculation of the natural frequency of the

s	n	κ	Measurements f	Δ	Calculations f (2.17)	f (viscous)	Δ	δ_i/δ_w	δ(meas.)/δ(calc.) HM	Martel *et al.*
1	0	1.841	4.65	1.4	4.68	4.66	1.34 (1.36)	0.19	1.05 (1.06)	1.02
2	0	3.054	6.32	1.8	6.35	6.33	1.70	0.37	1.06	1.03
0	1	3.832	6.84	1.2	6.85	6.84	0.92 (0.93)	1.64 (1.63)	1.30 (1.29)	1.26
3	0	4.201	7.80	2.2	7.84	7.81	2.04	0.59	1.08	1.04
4	0	5.318	9.26	2.4	9.29	9.27	2.39	0.83	1.01	0.97
1	1	5.331	8.57	1.5	8.60	8.59	1.43	2.01	1.05	1.03

TABLE 1. Measured and predicted frequencies and damping rates (non-dimensionalized so that $\Delta = 4a\delta/l_v$, $\delta = \delta_i + \delta_w$), the ratios of interior and boundary-layer damping, and the ratios of measured and predicted damping rates for modes with s nodal diameters and n nodal circles on HPLC water for a pinned contact line and surface cleaned as described in HM, § 5.1. Here, $f(2.17)$ is calculated from HM, (2.17), and $f(\text{viscous}) = f(2.17) - (\gamma/2\pi)$, where γ is the measured damping rate minus the calculated (theoretical) interior damping. The parenthetic numbers allow for finite depth, which proves to be marginally significant (within the present accuracy) only for the (1,0) and (0,1) modes. Martel *et al.*'s results allow for finite depth and incorporate the second-order boundary-layer correction.

s	n	κ	Measurements f	Δ	Calculations ($T = 62$) f (2.17)	f (viscous)	Δ	δ(meas.)/δ(calc.)
1	0	1.841	4.63	5.8	4.66	4.60	3.39	1.7
2	0	3.054	6.19	7.7	6.30	6.21	5.15	1.5
0	1	3.832	6.68	7.2	6.78	6.69	5.02	1.4
3	0	4.201	7.62	8.1	7.73	7.63	6.80	1.2
4	0	5.318	8.96	9.4	9.12	8.99	8.42	1.1
1	1	5.331	8.37	8.9	8.45	8.33	7.09	1.3

TABLE 2. Measured and predicted frequencies and damping rates (non-dimensionalized so that $\Delta = 4a\delta/l_v$, $\delta = \delta_i + \delta_w$) and the ratios of measured and predicted damping rates for modes with s nodal diameters and n nodal circles on filtered, distilled water for a pinned contact line and surface contaminated as described in HM, § 5.2. Here, $f(2.17)$ is calculated from HM, (2.17), and $f(\text{viscous}) = f(2.17) - (\gamma/2\pi)$, where γ is the measured damping rate minus the calculated (theoretical) interior damping.

(0,1) mode, which should be $f_{01} = 6.852$ Hz (in place of 6.753 Hz). Viscous correction reduces this to 6.84 Hz, which agrees with our measured value of 6.84 Hz.

Results for our contaminated-surface (HM, § 5.2) experiments are listed in table 2. The unexplained residual damping in these results (in contrast to the non-axisymmetric, clean-surface results) is significant within the experimental error.

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